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Engwerda, J.C.

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Stabilizability and Detectability of Discrete-Time Time-Varying Systems

J. C. ENGWERDA

Abstract—**In this note we introduce some notions of stabilizability and detectability for discrete time-varying systems. These concepts are introduced by making state-space decompositions of the system. This allows an intuitive interpretation and shows the difficulties which occur if one tries to derive general stabilizability and detectability properties.**

Moreover, we show by means of a counterexample that the notions of uniform stabilizability and uniform detectability, as defined by Anderson and Moore, do not imply stabilizability, respectively, detectability of the system.

I. Introduction

In the theory of linear time-varying difference equations the concept of "uniform asymptotic stability" plays an important role. This, since according to a theorem of Poincaré-Bendixson, uniform asymptotic stability of the linearized system implies uniform asymptotic stability of the nonlinear system. Now, J. L. Willems proved in [8, theorem 7.5.2] that the flow of a linear discrete time-varying system is uniformly asymptotically stable if and only if it is exponentially stable.

Therefore, a natural question is under which conditions such a system is exponentially stabilizable.

Stated differently, under which conditions does there exist a control sequence in the form of a state feedback, such that the resulting closed-loop system becomes exponentially stable.

For time-invariant systems these conditions are well known. For time-varying systems, however, this question is more complicated, and a general theory about it is lacking.

Hager and Horowitz [6] and Anderson and Moore [2] took a lead with the introduction of sufficient conditions for detectability and stabilizability of discrete-time time-varying systems. Moreover, they used these to solve some control and filtering problems. However, unfortunately the claim of Anderson and Moore that the time-varying discrete-time Kalman filter is exponentially stable under the conditions of uniform stabilizability and uniform detectability, is incorrect. We provide a counterexample to this claim here. Furthermore, we believe that the definitions which Hager and Horowitz give as those of Anderson and Moore do not give a clear insight into the basic underlying structural problems.

To obtain a better insight in these problems, the state-space decomposition approach given by Ludyck in [7] seems to be a more promising one. Therefore, we extend that analysis in this note.

Based on two state-space decompositions we discern several types of stabilizability and detectability. These concepts can be used to solve, e.g., the linear quadratic regulator problem, as will be reported elsewhere.

Since the proofs given in Ludyck [7] to obtain the state-space decompositions are not entirely correct, we also provide correct proofs of them.

The outline of the note is as follows.

First, in Section II we introduce some definitions and provide the counterexample. Then, by making a decomposition of the state-space at any time into three orthogonal subspaces, we obtain in Section III an equivalent system representation from which easily various stabilizability properties of the original system can be deduced. The decomposition originates from considering the reachability and exponential stability subspaces. In Section IV an analogous analysis is performed for detectability. Here the decomposition of the state-space into the unobservable subspace and its complement plays an important role. At last we combine the results in Section V in which we give a state-space description based on a simultaneous decomposition of the space into the reachable and unobservable parts. The note ends with some concluding remarks.

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The author is with Tilburg University, Tilburg, The Netherlands.
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II. Stabilizability

In this note we consider a system described by the following linear discrete-time time-varying recurrence equation:

$$\Sigma_y: \quad \begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k);\ x(k_0) = x \\ y(k) &= C(k)x(k). \end{aligned}$$

Here $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in \mathbb{R}^m$ the applied control, and $y(k) \in \mathbb{R}^p$ the output at time k. Moreover, we assume that all matrices $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are bounded. We use the following notation.

Notation 0

$v^T(i)$ denotes the transpose for $v(i)$.

$$v[k, l] := (v^T(k), \cdots, v^T(l))^T.$$

$$v[k, \cdot] := (v^T(k), v^T(k+1), \cdots)^T.$$

Im A denotes the image of the mapping defined by matrix A; Ker A denotes its kernel.

$$A(k+i, k) := A(k+i-1) * \cdots * A(k),\ i \geq 1, \text{ and } A(k, k) := I.$$

$$S[k, k-N] := [B(k) | A(k+1, k)B(k-1) | \cdots | A(k+1, k+1-N)B(k-N)].$$

$$W[N, N+i] := [C^T(N) | \{C(N+1)A(N+1, N)\}^T | \cdots | \{C(N+i)A(N+i, N)\}^T].$$

$O_{p,q}$:= zero matrix with p rows and q columns.

$x(k, k_0, x_0, u)$ is the state of the system at time k resulting from the initial state x_0 at time k_0 when the input $u[k_0, k-1]$ is applied.

$$y(k, k_0, x_0, u) := C(k)x(k, k_0, x_0, u). \qquad \square$$

If in Σ_y, $C(k)$ equals the identity matrix at any time k (i.e., we have full state observations), the subscript y is dropped.

We start our analysis by giving formal definitions of several notions of stability and stabilizability. In these definitions we use the concept of exponential convergence of a sequence $u[k_0, \cdot]$. This is defined as follows. We say that $u(\cdot)$ converges exponentially fast to zero if there exist positive constants α and M such that $\| u(k) \| < Me^{-\alpha(k-k_0)}$ for all $k > k_0$.

Definition 1: The initial state x of the system Σ_y is said to be stable at k_0 if $\lim_{k\to\infty} x(k, k_0, x, 0) = 0$; exponentially stable at k_0 if there exist positive constants α and M such that

$$\|x(k, k_0, x, 0)\| \leq Me^{-\alpha(k-k_0)} \|x\| \qquad \text{for any } k > k_0;$$

stabilizable at k_0 if there exists a control sequence $u[k_0, \cdot]$, with the property that $u(\cdot) \to 0$, such that $\lim_{k\to\infty} x(k, k_0, x, u) = 0$; exponentially stabilizable at k_0 if there exists a control sequence $u[k_0, \cdot]$, with the property that $u(\cdot)$ converges exponentially fast to zero, and positive constants α and M such that

$$\|x(k, k_0, x, u)\| \leq Me^{-\alpha(k-k_0)}\|x\| \qquad \text{for any } k > k_0.$$

The system Σ_y is called stable (respectively, exponentially stable, stabilizable, exponentially stabilizable) at k_0 if any initial state of Σ_y possesses the corresponding property at k_0.

As announced in the Introduction, we give in this section a counterexample for a result obtained by Anderson and Moore in [2].

To that end we first introduce their concepts of uniform stabilizability and uniform detectability and quote their Corollary 5.4.

Definition 2: Σ_y is uniformly stabilizable if there exist integers $s, t \geq 0$ and constants d, b with $0 \leq d < 1, 0 < b < \infty$, such that whenever

$$\|A(k+1, k+1-t)v\| \geq d\|v\|$$

for some v, k, then

$$v^T S[k, k-s]S^T[k, k-s]v \geq bv^T v.$$

Σ_y is uniformly detectable if there exist integers $s, t \geq 0$ and constants d, b with $0 \leq d < 1, 0 < b < \infty$, such that whenever

$$\|A(k+t, k)v\| \geq d\|v\|$$

for some v, k, then

$$v^T W[k+s, k]W^T[k+s, k]v \geq bv^T v. \qquad \square$$

"Corollary 3" [2, Corollary 5.4]: If Σ_y is uniformly detectable, there exists a bounded sequence $K(k)$ such that $x(k+1) = (A(k) - K(k)C(k))x(k)$ is exponentially stable. $\square$

This corollary is an immediate consequence of Theorem 5.3 in the above-mentioned paper.

In this theorem it is claimed that if the system is uniformly stabilizable and uniformly detectable, then the Kalman filter is exponentially stable (under the usual system noise assumptions).

Now, consider the following example.

Counterexample 4: Let

$$A(2k) = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}; \; C(2k) = (0 \quad 1); \; A(2k+1) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix};$$

$$C(2k+1) = (0 \quad 0), \; k = 0, 1, 2, \cdots.$$

Then, with $s = 0$, $t = 1$, $d = \frac{3}{4}$, $b = \frac{3}{64}$, we see that Σ_y is uniformly detectable.

However, if we consider the initial state $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ at time zero, we see that for any sequence $K(\cdot)$, $x(k+1) = (A(k) - K(k)C(k))x(k)$, $x(0) = x$ is not stable. Which contradicts Corollary 3. $\square$

A direct implication of this example is that the above-mentioned Theorem 5.3 is incorrect too. By dualizing this example, i.e., take $A(\cdot) = A^T(\cdot)$ and $B(\cdot) = C^T(\cdot)$, we see that the uniform stabilizability condition is not sufficient either to conclude that there exists a state feedback such that the closed-loop system becomes (exponentially) stable.

Since the major reason for introducing the uniform stabilizability and detectability condition is to have a criterion from which exponential stabilizability, respectively, exponential detectability of the system can be concluded, we concentrate in the rest of this note on finding such criteria.

III. Sufficient Conditions for Exponential Stabilizability

In this section we derive sufficient conditions for exponential stabilizability of Σ.

To that end we first consider the exponential stability subspace at k_0, denoted by $X_e^-(A(\cdot, k_0))$.

This subspace consists of all initial states at k_0 which are exponentially stable. Similarly, we define the stability (or modal) subspace at k_0, denoted by $X^-(A(\cdot, k_0))$.

That $X^-(A(\cdot, k_0))$ and $X_e^-(A(\cdot, k_0))$ are indeed linear subspaces is easily verified. The next lemma tells us, moreover, that the exponential stability subspace is $A(k)$-invariant. This property is used later on in this section to make an appropriate state-space decomposition.

Lemma 5:

$$A(k)X_e^-(A(\cdot, k)) \subset X_e^-(A(\cdot, k+1)).$$

Proof: Consider Σ at time k. Let x be an exponentially stable state. Then, we have by definition that for some positive constants α and M

$$\|x(t+1, k, x, 0)\| \leq Me^{-\alpha(t+1-k)}\|x\|$$

at any time $t \geq k$. But, since $x(t+1, k, x, 0) = x(t+1, k+1, A(k)x, 0)$ we obtain immediately that $A(k)x \in X_e^-(A(\cdot, k+1))$, which proves the lemma. $\square$

Another property that plays an important role in our analysis concerns the reachability subspace. Its definition reads as follows.

Definition 6: The state x is said to be reachable at k_0 from zero if there exists a control sequence $u[N, k_0 - 1]$ with $-\infty < N < k_0$ such that $x(k_0, N, 0, u) = x$.

The subspace consisting of all reachable states from zero at k_0 is called the reachability subspace at k_0, and denoted by R_{k_0}. Its dimension is denoted by r_{k_0}.

The reachability property we are interested in is stated in the next lemma. A formal proof can be found in Engwerda [3].

Lemma 7: Consider time k_0, and define $A(k) = 0$ and $B(k) = 0$ for $k < k_0$. Then at any time $t \geq k_0$ we have

$$A(k)R_k + \operatorname{Im} B(k) = R_{k+1}. \qquad \square$$

This lemma tells us in particular that the reachability subspace is also $A(k)$-invariant.

Now consider the following state-space decomposition:

$$X_1(k) = R_k \cap X_e^-(A(\cdot, k));$$

$$X_2(k) \oplus X_1(k) = R_k;$$

$$X_3(k) \oplus X_2(k) \oplus X_1(k) = \mathbb{R}^n$$

where X_1, X_2, and X_3 are chosen orthogonal.

With respect to a basis adapted to this state-space decomposition the next important corollary holds (see Lemmas 5 and 7).

Corollary 8: There exists an orthonormal state transformation $T'(\cdot)$ such that with $x(k) = T'(k)x'(k)$, Σ is described by

$$\Sigma_1': \begin{pmatrix} x_1'(k+1) \\ x_2'(k+1) \\ x_3'(k+1) \end{pmatrix} = \begin{pmatrix} A_{11}'(k) & A_{12}'(k) & A_{13}'(k) \\ 0 & A_{22}'(k) & A_{23}'(k) \\ 0 & 0 & A_{33}'(k) \end{pmatrix} \cdot \begin{pmatrix} x_1'(k) \\ x_2'(k) \\ x_3'(k) \end{pmatrix} + \begin{pmatrix} B_1'(k) \\ B_2'(k) \\ B_3'(k) \end{pmatrix} u(k)$$

where

Σ_1': $x_1'(k+1) = A_{11}'(k)x_1'(k) + B_1'(k)u(k)$ is exponentially stable at any time $k \geq k_0$.

Σ_2': $x_2'(k+1) = A_{22}'(k)x_2'(k) + B_2'(k)u(k)$ is reachable at any time $k \geq k_0$, and is not exponentially stable.

$$\Sigma_3': x_3'(k+1) = A_{33}'(k)x_3'(k).$$

An important property of this transformed system is that it preserves the convergence properties of the base system Σ (provided $T'(\cdot)$ is chosen orthonormal). That is, $x'(k) \to 0$ if and only if $T'(k)x'(k) \to 0$. So, instead of analyzing the convergence properties of Σ we can concentrate in the sequel on those of Σ'.

In the remainder of this section we derive sufficient conditions in terms of the transformed system for exponential stabilizability of Σ.

From Corollary 8 we immediately have the following result.

Lemma 9: Consider the transformed system (1).

If Σ is exponentially stabilizable at k_0, then Σ_2' has to be exponentially stabilizable at k_0 and Σ_3' is exponentially stable at k_0. □

In fact, we can conclude much more from the state-space decomposition.

We see namely that all disturbances entering the system Σ_2' at any time $k \geq k_0$ in a specific way [namely via Im $A_{23}'(k)A_{33}'(k, k_0)$] can be controlled exponentially fast to zero.

We investigate this phenomenon in more detail now.

Definition 10: Consider Σ_d: $x(k+1) = A(k)x(k) + B(k)u(k) + G(k)d(k)$, where $d(\cdot)$ is a known disturbance and $x(k_0) = x$.

Then Σ_d is called exponentially disturbance stabilizable at k_0 if for any initial state at k_0 and for any disturbance there exists a control sequence $u[k_0, \cdot]$ converging exponentially fast to zero, such that $x(k, k_0, x, u, d)$ converges exponentially fast to zero. Here $x(k, k_0, x, u, d)$ is defined similar to $x(k, k_0, x, u)$. □

From the above considerations we have the following theorem.

Theorem 11: Σ is exponentially stabilizable at k_0 iff the following two conditions are satisfied at k_0:

i) Σ_3' is exponentially stable at k_0;

ii) Σ_{2d}': $x_2'(k+1) = A_{22}'(k)x_2'(k) + B_2'(k)u(k) + A_{23}'(k)A_{33}'(k, k_0)d$ is exponentially disturbance stabilizable at k_0.

Proof: That both the conditions are necessary was argued in Lemma 9 and the ensuing remark.

That they are also sufficient is seen as follows. Due to assumption ii) we know that for any $x_3'(k_0)$ there exists a control sequence $\bar{u}(\cdot)$, which converges exponentially fast to zero, such that the second state component of $x'(k, k_0, x', \bar{u})$ converges exponentially fast to zero. Here $x' := (x_1', x_2', x_3')^T$.

But, since Σ_1': $x_1'(k+1) = A_{11}'(k)x_1'(k)$ is exponentially stable at any time $k \geq k_0$, and matrix B is bounded, this implies that the first state component of $x'(k, k_0, x', \bar{u})$ converges also exponentially fast to zero. As the third state component of $x'(k, k_0, x', u)$ converges exponentially fast to zero irrespective of what u is, it is clear now, that with $u = \bar{u}$, we have found an appropriate control sequence which stabilizes Σ' exponentially fast. □

So, the main problem left to be solved is to give conditions under which Σ_{2d}' is exponentially disturbance stabilizable.

We provide sufficient conditions. Therefore, we introduce the concept of periodic smooth controllability, as defined by Engwerda in [3]. Roughly spoken, a system is called periodically smoothly controllable if there exists a finite time period such that whenever such a time period has passed, the system has been at least once controllable during that period. Formally its definition reads as follows (for the definition of S see notation 0).

Definition 12: Σ is called periodically smoothly controllable at k_0 if there exist constants ϵ, k_1, and N such that for all $k > 0$ there exists an integer $k_2(k)$ in the interval $(k_0 + (k-1)*k_1, k_0 + k*k_1)$ for which $S[k_2 - N, k_2]S^T[k_2 - N, k_2] \geq \epsilon I$. □

Note that without loss of generality we can take $N = 2*k_1$ in this definition since, whenever $N < 2*k_1$, we have that $S[k_2 - N, k_2]S^T[k_2 - N, k_2] \geq \epsilon I$ implies that the same inequality holds for $S[k_2 - 2*k_1, k_2]S^T[k_2 - 2*k_1, k_2]$.

Theorem 13: Consider Σ_{2d}' from Theorem 11.

Let Σ_2' be periodically smoothly controllable at k_0 and Σ_3' exponentially stable at k_0. Then, Σ_{2d}' is exponentially disturbance stabilizable at k_0.

Proof: First, we note that due to the exponential stability assumption on Σ_3', $A_{23}'(k)A_{33}'(k, k_0)d$ converges exponentially fast to zero.

Now consider the time interval (k_0, k_1).

Let $e(k_2(1))$ denote the sum of all disturbances entering Σ_{2d}' during this time period, i.e.,

$$e(k_2(1)) = \sum_{i=k}^{k_2(1)} A_{22}'(k_2(1), i)A_{23}'(i)A_{33}'(i, k_0)d.$$

Consider the input

$$u[k_0, k_2(1) - N - 1] = 0, \text{ and}$$
$$u[k_2(1) - N, k_2(1)] = -S_2^{\prime T}(S_2'S_2^{\prime T})^{-1}(e(k_2(1)) + x_2')$$

where $S_2' := S[k_2(1), k_2(1) - N]$ w.r.t. Σ_2' and x_2' is the initial state of Σ_{2d}'.

With this input, $x_2'(k_2(1) + 1)$ becomes zero.

We show now by induction that it is possible to regulate $x_2'(k_2(k) + 1)$ to zero for any k. Let therefore t be any integer greater than one.

Consider the interval $(k_0 + (t-2)*k_1, k_0 + t*k_1)$. The sum of all exogenous influences entering the reachable subsystem via matrix $A'(\cdot)$ from $k_2(t-1) + 1$ until $k_2(t)$ on is then

$$e(k_2(t)) := \sum_{i=k_2(t-1)+1}^{k_2(t)} A_{22}'(k_2(t), i)A_{23}'(i)x_3'(i).$$

Since by induction hypothesis $x_2'(k_2(t) + 1)$ is zero, application of the input

$$u[k_2(t-1) + 1, k_2(t) - N - 1] = 0, \text{ and}$$
$$u[k_2(t) - N, k_2(t)] = -S_2^{\prime T}(S_2'S_2^{\prime T})^{-1}e(k_2(t))$$

yields that $x_2'(k_2(t) + 1) = 0$. Here $S_2' := S[k_2(t), k_2(t) - N]$ w.r.t. Σ_2'.

This completes the induction argument.

Moreover, we observe that $\|u(k)\| \leq M\|e(k_2(t))\|$ for some constant M, since $S_2'S_2^{\prime T} \geq \epsilon I$ and S_2' is bounded. Therefore, $\|x_2'(k)\| \leq M'\|e(k_2(t))\|$ for all $k \in (k_2(t-1), k_2(t))$. Due to the exponential convergence of $e(k_2(t))$ to zero when t tends to infinity, we conclude that both $x_2'(k)$ and $u(k)$ converge exponentially fast to zero when k tends to infinity. This completes the proof. □

Corollary 14: Σ is exponentially stabilizable at k_0 if:

i) Σ_3' is exponentially stable at k_0;

ii) Σ_2' is periodically smoothly controllable at k_0. □

IV. Sufficient Conditions for Exponential Detectability

We now give a necessary and sufficient condition for exponential detectability of Σ_y. To that end we first introduce the notion of observability.

Definition 15: The initial state x of the system Σ_y is said to be unobservable at k_0 if $y(k, k_0, x, 0) = 0$ for all $k \geq k_0$.

The set of all unobservable states at k_0 is denoted by U_{k_0} and called the unobservable subspace. Σ_y is said to be observable at k_0 if $x = 0$ is the only unobservable state of Σ_y at k_0. □

Remark: Note that Σ_y is observable at k_0 iff $U_{k_0} = 0$! □

Analogous to Lemmas 4 and 5 we have that U_k is $A(k)$-invariant. This is the content of Lemma 16.

Lemma 16:

$$A(k)U_k \subset U_{k+1}.$$

Proof: Let x_1 be an element of $A(k)U_k$. By definition there exists then an x_0 such that:

i) $x_1 = A(k_0)x_0$; and

ii) $y(k, k_0, x_0, 0) = 0$ for all $k \geq k_0$.

The rest of the proof follows now from the observation that $y(k, k_0 + 1, x_1, 0) = y(k, k_0, x_0, 0)$. □

We define now the notion of (exponential) detectability.

Definition 17: The initial state x of Σ_y is said to be detectable at k_0 if there exists a finite integer $N > 0$ such that $x(k_0)$ modulo $X^-(A(\cdot, k_0))$ is determined from any $y[k_0, k_0 + N - 1]$ and $u[k_0, k_0 + N - 2]$. Σ_y is called detectable at k_0 if all states x are detectable at k_0. Σ_y is called exponentially detectable at k_0 if in the above definition of detectability $X^-(A(\cdot, k_0))$ is replaced by $X_e^-(A(\cdot, k_0))$. □

Next, consider the state-space decomposition

$$X_1(k) = X_e^-(k) \cap U_k,$$
$$X_2(k) \oplus X_1(k) = U_k,$$
$$X_3(k) \oplus X_2(k) \oplus X_1(k) = \mathbb{R}^n$$

where X_1, X_2, and X_3 are chosen orthogonal.

With respect to a basis adapted to this decomposition, the following analog of Corollary 8 holds.

Corollary 17: There exists an orthogonal state-space transformation $x(\cdot) = T''(\cdot)x''(\cdot)$ which does not affect the boundedness property of the system parameters such that Σ_y is described by the recurrence equation

$$\Sigma_y'': \begin{pmatrix} x_1''(k+1) \\ x_2''(k+1) \\ x_3''(k+1) \end{pmatrix} = \begin{pmatrix} A_{11}''(k) & A_{12}''(k) & A_{13}''(k) \\ 0 & A_{22}''(k) & A_{23}''(k) \\ 0 & 0 & A_{33}''(k) \end{pmatrix} \cdot \begin{pmatrix} x_1''(k) \\ x_2''(k) \\ x_3''(k) \end{pmatrix} + \begin{pmatrix} B_1''(k) \\ B_2''(k) \\ B_3''(k) \end{pmatrix} u(k)$$

$$y(k) = (0 \quad 0 \quad C_3''(k))x''(k)$$

where

Σ_1'': $x_1''(k+1) = A_{11}''(k)x_1''(k)$,

is exponentially stable at any time $k \geq k_0$.

Σ_3'': $x_3''(k+1) = A_{33}''(k)x_3''(k) + B_3''(k)u(k)$;

$y(k) = C_3''(k)x''(k)$, is observable at any time $k \geq k_0$. □

With the notation from the previous corollary we have the following.

Theorem 18: Σ_y is exponentially detectable at k_0 iff Σ_2'': $x_2''(k+1) = A_{22}''(k)x_2''(k)$ is exponentially stable at k_0.

Proof:

"$\Rightarrow$" Consider the transformed system Σ_y''.

Since Σ_y is exponentially detectable, the inclusion $X_2 \subset X_e^-(A(\cdot, k_0))$ must hold. Consequently, Σ_2'' has to be exponentially stable at k_0.

"$\Leftarrow$" From Corollary 17 we know that Σ_1'' is exponentially stable at any time $k \geq k_0$. Due to the assumption that Σ_2'' is exponentially stable at k_0 we have that the following system

$$\begin{pmatrix} x_1''(k+1) \\ x_2''(k+1) \end{pmatrix} = \begin{pmatrix} A_{11}''(k) & A_{12}''(k) \\ 0 & A_{22}''(k) \end{pmatrix} \begin{pmatrix} x_1''(k) \\ x_2''(k) \end{pmatrix}$$

is also exponentially stable at k_0.

So, $X_1 \oplus X_2 \subset X_e^-(A(\cdot, k_0))$.

Since Σ_3'' is observable at k_0, it is clear now that Σ_y'', and therefore Σ_y is exponentially detectable at k_0. □

V. Sufficient Conditions for Simultaneously Exponentially Stabilizable and Detectable Systems

In the previous two sections we gave necessary and sufficient conditions for exponentially stabilizable and exponentially detectable systems, respectively. In the present section we combine these results.

We derive now sufficient conditions to conclude that the system Σ_y is both exponentially stabilizable and exponentially detectable.

To that end we again make a state-space decomposition.

Consider

$$X_1(k) = R_k \cap U_k;$$

$$X_2(k) \oplus X_1(k) + R_k;$$

$$X_3(k) \oplus X_2(k) \oplus X_1(k) = \mathbb{R}^n$$

where X_1, X_2, and X_3 are chosen again orthogonal.

Then, analogous to Corollaries 8 and 17 we have the following.

Corollary 19: There exists an orthogonal state-space transformation $x(\cdot) = T`(\cdot)x`(\cdot)$ which does not affect the boundedness property of the system parameters such that Σ_y is described by the recurrence equation

$$\Sigma`_y: \begin{pmatrix} x`_1(k+1) \\ x`_2(k+1) \\ x`_3(k+1) \end{pmatrix} = \begin{pmatrix} A`_{11}(k) & A`_{12}(k) & A`_{13}(k) \\ 0 & A`_{22}(k) & A`_{23}(k) \\ 0 & 0 & A`_{33}(k) \end{pmatrix} \cdot \begin{pmatrix} x`_1(k) \\ x`_2(k) \\ x`_3(k) \end{pmatrix} + \begin{pmatrix} B`_1(k) \\ B`_2(k) \\ B`_3(k) \end{pmatrix} u`(k)$$

$$y(k) = (0 \quad C`_1(k)C`_2(k))x`(k)$$

where

$\Sigma`_1$: $x`_1(k+1) = A`_{11}(k)x`_1(k) + B`_1(k)u`(k)$

is reachable at any time $k \geq k_0$;

$\Sigma`_2$: $x`_2(k+1) = A`_{22}(k)x`_2(k) + B`_2(k)u`(k) + A`_{23}(k)A`_{33}(k+1, k_0)d$

$y(k) = C`_2(k)x`_2(k)$ is both reachable and observable at any time $k \geq k_0$;

$\Sigma`_3$: $x`_3(k+1) = A`_{33}(k)x`_3(k)$. □

Theorem 20: With the notation as in Corollary 19 we have that $\Sigma`_y$ is both exponentially stabilizable and exponentially detectable at k_0 if the following three conditions are satisfied:

i) $\Sigma`_1$ is exponentially stable for all $k \geq k_0$;
ii) $\Sigma`_2$ is exponentially disturbance stabilizable and observable at k_0;
iii) $\Sigma`_3$ is exponentially stable at k_0.

Proof: That $\Sigma`_y$ is exponentially stabilizable at k_0 under these conditions is proved similarly to the proof of Theorem 11.

To prove exponential detectability of $\Sigma`_y$ at $k_0$, we note that conditions i) and iii) imply that $X_1 \oplus X_3 \subset X_e^-(A`(\cdot, k_0))$.

Using this and the property that any state from X_2 can be observed, we have that consequently any element of the factor space $\mathbb{R}^n$ modulo $X_e^-(A`(\cdot, k_0))$ can be observed. So, $\Sigma`_y$ is exponentially detectable at k_0. □

Note that condition ii) in this theorem is also a necessary condition, and that, moreover, exponential stability of $\Sigma`_1$ and $\Sigma`_3$ at k_0 are also necessary requirements.

VI. Concluding Remarks

In this note we showed that exponential stabilizability and exponential detectability properties of a system can be analyzed like in the time-invariant case by using appropriate state-space decompositions.

On the one hand this is due to our choice of the definitions of stabilizability and detectability. In our definition of stabilizability we required, namely, that additional to the property that the closed-loop system must be stable after a well-chosen input has been applied, the input itself must be stable too.

On the other hand this is due to our choice of the state-space decompositions. They are all chosen in such a way that the convergence properties of the transformed and original system remain the same.

A direct consequence of this last mentioned prerequisite was that, when we analyzed systems which are both stabilizable and detectable, we did not choose the state-space decomposition which seems at first glance to be the most appropriate one for analyzing these systems.

Taking into consideration the several attempts which have been taken in the past to analyze stabilizability and detectability aspects of time-varying systems and the relative ease by which results are obtained when using this analysis, it seems worthwhile to deepen this analysis in the future.

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A New Class of Shift-Varying Operators, Their Shift-Invariant Equivalents, and Multirate Digital Systems

DAVID G. MEYER

Abstract—**We define a new class of linear, shift-varying operators that generalizes the notion of N-periodicity. It is shown that shift-invariant equivalents for these new operators exist, and that the equivalence is in a strong sense—preserving both algebraic and analytic systems properties. We show that multirate sampled-data systems, although not generally periodic, fall into this new class. Kranc vector switch decomposition and block filter implementations for single-input single-output multirate systems are connected under the unifying framework of shift-invariant equivalents, and this framework is the way to extend them both to multiinput, multioutput systems.**

I. Introduction

Consider the shift-varying discrete-time system $y = Hu$ where the operator H is given by

$$y_i(l) = \sum_{k=-\infty}^{\infty} \sum_{j=1}^{m} h_{ij}(l, k)u_j(k) \qquad i = 1, 2, \cdots, p. \tag{1}$$

Here, of course,

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

is a p-vector of outputs and, similarly, u is an m-vector of inputs. As is usual, $h_{ij}(l, k)$ is called the *impulse response* (*matrix*) of the system and is causal if $h_{ik}(l, k) = 0$ for $l < k$ and each i and j.

The system described by $y = Hu$ with H given as in (1) is said to be N-periodic if a smallest positive integer N exists so that $h_{ij}(l+N, k+N) = h_{ij}(l, k)$ for all i, j, l, and k. Notice that a shift-invariant operator is 1-periodic; thus the notion of an N-periodic shift-varying operator is a generalization of the notion of a shift-invariant operator.

It is known [5], [7], [10], [11] that to every N-periodic operator of the form (1), it is possible to associate, in a canonical and strong sense, an equivalent mN-input, pN-output shift-invariant operator. This association is exploited in [7] to analyze properties of feedback systems that employ *periodic* compensation.

In this note we introduce a new class of shift-varying operators that generalizes the notion of N-periodic. We then show that it is possible to associate, in a canonical and strong sense, an equivalent shift-invariant operator with any member of this class and, as in the N-periodic case, the association may be used for analysis of these more general operators. Finally, we show that operators in this new class arise in an important and often encountered context—*multirate* digital systems—and that the theory of shift-invariant equivalents for these new operators is a unifying tool in the study of multirate systems.

II. The New Class

Definition 1: Given sets of positive integers $\{M_j\}_{j=1}^{m}$ and $\{P_i\}_{i=1}^{p}$, we say that the operator defined by (1) is (P_i, M_j)-shift-varying if, and only if,

$$h_{ij}(l + P_i, k + M_j) = h_{ij}(l, k) \qquad \forall\, l, k \tag{2}$$

and the M_j and P_i are the "smallest" positive integers that satisfy (2) in the sense that if $\{\tilde{M}_j\}_{j=1}^{m}$ and $\{\tilde{P}_i\}_{j=1}^{p}$ is another set of positive integers satisfying (2), then $M_j | \tilde{M}_j$ and $P_i | \tilde{P}_i$ for each i and j.

In words, H is (P_i, M_j)-shift-varying if shifting the jth input by M_j for each j does not change the ith output, for each i, except for a shift by P_i. Note that an N-periodic operator is precisely a (P_i, M_j)-shift-varying operator with $M_j = P_i = N$ for each i and j (and vice-versa). When the P_i and M_j are *not* all the same, however, it may be verified that a (P_i, M_j)-shift-varying operator is not N-periodic for any N. It is thus readily apparent that this new class of operators does, indeed, generalize the notion of N-periodicity.

For single-input/single-output systems, the property described in Definition 1 has been examined in [9] as it relates to multirate digital filtering. There it was noted that a digital filter with input sampling interval KT and output sampling interval LT was (K, L)-shift-varying and had a shift-invariant block realization. The algebraic and analytic properties of this correspondence were not investigated.

We will show the connection between multirate sampled data systems and (P_i, M_j)-shift-varying operators in Section IV.

III. Shift-Invariant Equivalents

For N-periodic m-input, p-output systems, it is possible to associate an mN-input and pN-output shift-invariant system. (See [7] or [10].) This is motivated by noting that the matrix representation (with respect to the standard basis) of the operator H given by (1) when $h_{ij}(l+N, k+N) = h_{ij}(l, k)$ has a $pN \times mN$ block Toeplitz structure:

$$M = \begin{bmatrix} M_0 & 0 & 0 & \cdots \\ M_1 & M_0 & 0 & \cdots \\ M_2 & M_1 & M_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and thus is also the matrix representation of a $pN \times mN$ shift-invariant operator. This shift-invariant equivalent has a transfer function given by $M(z) = \sum_{n=0}^{\infty} M_n z^{-n}$ in the usual fashion.

This result for the N-periodic case is true in the more general (P_i, M_j)-shift-varying case. We have the following.

Theorem 1: Let the operator H given by (1) be (P_i, M_j)-shift-varying. Define

$$P \triangleq \sum_{i=1}^{p} P_i \text{ and } M \triangleq \sum_{j=1}^{m} M_j.$$

Then there is an M-input, P-output shift-invariant equivalent operator associated with H.

Proof: Let E denote the space of real sequences; E^q denotes the Cartesian product of q copies of E. Denote the right shift operator on the jth component in E^q by Λ_j.

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The author is with the Department of Electrical Engineering, University of Virginia, Charlottesville, VA 22901.

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